Linear response theory for a pair of coupled one-dimensional condensates of interacting atoms

Linear response theory for a pair of coupled one-dimensional condensates of interacting atoms

Vladimir Gritsev,[1] Anatoli Polkovnikov,[2] and Eugene Demler[1]
[1]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[2]*Department of Physics, Boston University, Boston, Massachusetts 02215, USA*
(Received 19 January 2007; published 15 May 2007)

We use the quantum sine-Gordon model to describe the low-energy dynamics of a pair of coupled one-dimensional condensates of interacting atoms. We show that the nontrivial excitation spectrum of the quantum sine-Gordon model, which includes soliton and breather excitations, can be observed in experiments with time-dependent modulation of the tunneling amplitude, potential difference between condensates, or phase of tunneling amplitude. We use the form-factor approach to compute structure factors corresponding to all three types of perturbations.

DOI: 10.1103/PhysRevB.75.174511 PACS number(s): 03.75.Lm, 03.75.Kk, 03.75.Nt

I. INTRODUCTION

When discussing two-dimensional classical or one-dimensional quantum systems, no model has a wider range of applications than the sine-Gordon model. The sine-Gordon (SG) model was originally studied in the context of high-energy physics[1,2] and later became a prototypical model of low-dimensional condensed-matter systems. In statistical physics, it has been successfully applied to describe the Kosterlitz-Thouless transition in two-dimensional superconductors and superfluids[3,4] as well as commensurate to incommensurate transitions[5] on surfaces. The quantum sine-Gordon model was used to describe the superfluid to Mott transition of bosons in one dimension in such systems as Cooper pairs in Josephson junctions arrays[6] and cold atoms in optical lattices.[7] Sine-Gordon model also provides a useful framework for understanding properties of one-dimensional spin systems with easy axis and/or plane anisotropies or in the presence of magnetic field (see Ref. 8 for review). Interacting electron systems in one dimension often exhibit phase transitions between states with short-range and power-law correlations. Such transitions are also expected to be in the universality class of the quantum KT transition described by the quantum sine-Gordon model.[8] Large classes of boundary-related phenomena,[9] including the problem of an impurity in an interacting electron gas (the so-called quantum impurity model) (see Ref. 10 for review), Josephson junctions with dissipation,[10] Kondo-like models,[11] and single electron transport in ultrasmall tunnel junctions, have also been discussed in the framework of the boundary sine-Gordon model. Quantum sine-Gordon model provides a universal description for such wide range of systems because it is the simplest model with a gapped spectrum and relativistic low-energy dispersion.

One approach to understand the sine-Gordon model is based on the renormalization-group analysis (see Refs. 8 and 12 for the review). This method has been successfully applied to discuss thermodynamic and transport properties of two-dimensional superfluids and superconductors.[3]

Another more detailed approach is based on the integrability of the model and existence of the complete exact solution.[13,14] From this analysis, we know that the excitation spectrum of the model consists of solitons, antisolitons, and their bound-state breathers. The number and properties of different breathers are controlled by the interaction strength (related to the parameter γ introduced in Sec. II A below).

While theoretical understanding of the quantum SG model is quite advanced, very few experimental studies have been done that addressed dynamics of this system. Several experimental studies on one-dimensional magnets interpreted resonances in neutron-scattering and ESR experiments as breather-type excitations.[15–17] All these examples include spin-$1/2$ magnetic chains perturbed by an effective g tensor and the Dzyaloshinskii-Morya interaction or by a staggered field in copper benzoate[18,19] and dimethylsulfoxide[20] and/or by strong magnetic field, like in copper pyrimidine dinitrate.[17] Apart from these measurements, we do not have experimental evidences to verify our understanding of the spectrum of the quantum SG model. Considering the importance of the sine-Gordon model for understanding one-dimensional quantum systems, it is of great interest to find new, more direct approaches for experimental investigation of this fundamental model.

In this paper, we show that a pair of coupled one-dimensional condensates provides realization of the quantum sine-Gordon model. Such systems were recently realized in cold atoms in microtraps, where the rf potential controls the tunneling between the condensates.[21–23] The advantage of using cold atoms to realize the quantum SG model is that cold atom systems are highly tunable. For example, in Refs. 24–26 it was demonstrated that the interaction and thus the Luttinger parameter describing one-dimensional condensates can be varied in a very wide range. Similarly, the tunneling between the two condensates, which controls the gap in the sine-Gordon theory (see details below), can be tuned to a high precision. Coupled condensates can also be realized in optical lattices using superlattice potentials.[21,27]

One possibility to probe the excitation spectrum of the quantum sine-Gordon model is to study the response of the system to small periodic modulations of various parameters of the model. In this paper, we discuss three types of such modulation experiments: modulation of the tunneling amplitude, modulation of the potential difference between the condensates, and modulation of the phase of the tunneling amplitude (see Fig. 1). By observing the frequency dependence of the absorbed energy in various modulation experiments, one can measure structure factors associated with appropriate

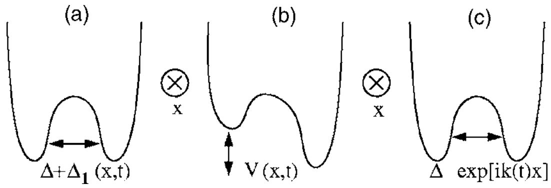

FIG. 1. Three types of experimental setups proposed to study linear response in two coupled condensates. The figure shows a projection of the confinement potential. In setup a) the tunneling amplitude is modulated: $\Delta(x,t)=\Delta+\Delta_1(x,t)$ (see Eqs. (8) and (10)); setup b) corresponds to the space-time-dependent population imbalance accessed via the change $V(t)$ of the confinement depth in one of the well; in experiment of type c) the space-time modulation comes from winding of the phase of single particle tunneling, $\Delta(x,t)=\Delta\exp(ik(t)x)$. Δ is parameter defined in Eq. (8). The x-direction coincides with the longitudinal direction of condensates and is out of plane of the figure.

perturbations. Here we theoretically compute structure factors for all three types of modulations (see Figs. 2–5). As we demonstrate below, different types of modulations expose different parts of the quantum sine-Gordon spectrum and provide complementary information about the system. We note that the calculations of structure factors for several types of operators in the quantum sine-Gordon model were done in earlier papers in the context of condensed-matter systems,[28–30] mainly to describe properties of Mott insulating states. These results were summarized in a recent review.[31] Somewhat similar analysis has been done in the context of the boundary sine-Gordon model (see Ref. 10 for the review). In this paper, we apply and extend this earlier analysis.

We note in passing that in a separate publication, we discussed a different possibility to study the structure of the sine-Gordon model using quench experiments.[32] There, we showed that the power spectrum of the phase oscillations after a sudden split of a single condensate into two contains detailed the information about various excitations in the sine-Gordon model. The quench and modulation experiments are thus complementary ways to extract this information.

This paper is organized as follows. First, we give an effective description of the system in Sec. II A. Then, we briefly summarize general facts about the quantum sine-Gordon model in Sec. II B. In Sec. III, we outline the explicit construction of the sine-Gordon form factors. In Sec. IV, we evaluate and analyze the structure factors for different kinds of perturbation. Finally, in Sec. V, we discuss experimental implications and extensions of our work and summarize the results.

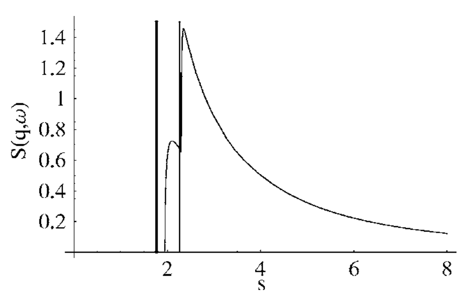

FIG. 2. Structure factor for the experiment of type (a) corresponding to the phase operator $\cos(\beta\phi)$ for the Luttinger parameter $K=1.15$ and the tunneling gap $\Delta=0.1$. Peaks correspond to the following leading contributions (from left to right): a single breather B_2, breathers B_1B_1, and soliton-antisoliton A_+A_-.

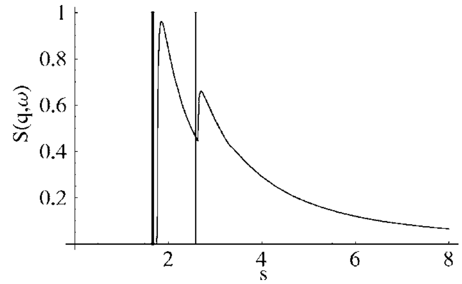

FIG. 3. Structure factor for the experiment of type (a) corresponding to the phase operator $\cos(\beta\phi)$ for $K=1.4$ and tunneling $\Delta=0.1$. Peaks correspond to the following leading contributions (from left to right): single breather B_2, breathers B_1B_1, single breather B_4, and soliton-antisoliton A_+A_-.

II. MICROSCOPIC MODEL

A. Effective model of two coupled condensates

We consider a system of two coupled interacting one-dimensional condensates. If interactions between atoms are short ranged, the Hamiltonian providing a microscopic description of the system is

$$H=\sum_{j=1,2}\int dx\left[-\frac{\hbar^2}{2m}\partial_x\Psi_j^\dagger(x)\partial_x\Psi_j(x)+gn_j^2(x)\right]$$
$$-t_\perp\int dx(\Psi_1^\dagger(x)\Psi_2(x)+\Psi_2^\dagger(x)\Psi_1(x)), \qquad (1)$$

where $n_j(x)=\Psi_j^\dagger(x)\Psi_j(x)$. Here, $t_\perp$ is the coupling which characterizes the tunneling strength between the two systems, and g is the interaction strength related to the three-dimensional (3D) scattering length a_{3D} and the transverse confinement length $a_\perp=\sqrt{\hbar/m\omega_\perp}$ via[33]

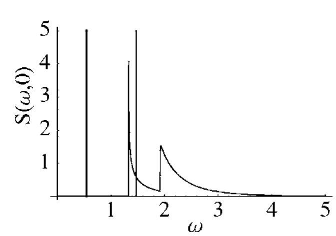

FIG. 4. Structure factor $S(w,q=0)$ for the setup scheme of type (b) corresponding to operator $\mathcal{O}=\partial_t\phi$ at $K=1.6$ and $\Delta=0.05$. The δ peaks (solid bold line) corresponding to the breathers B_1 (thick line) and B_3 (thin line) are coherent contributions, whereas peaks in the incoherent background (from left to right) correspond to the breather B_1-B_2 and soliton-antisoliton contributions.

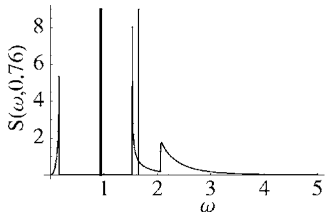

FIG. 5. Structure factor $S(w,q=0.76)$ for the setup scheme of type (b) corresponding to operator $\mathcal{O}=\partial_t\phi$ at $K=1.6$ and $\Delta=0.05$. The structure of peaks is the same as in Fig. 4 except for the appearance of the satellite breathers peak at small ω. It starts from nonzero q and exists further for all q's.

$$g \approx \frac{4\hbar^2}{ma_\perp}\frac{a_{3\mathrm{D}}}{a_\perp}\left(1 - C\frac{a_{3\mathrm{D}}}{a_\perp}\right)^{-1}, \tag{2}$$

where $C=1.4603\cdots$. In what follows, we assume that we are far from the confinement-induced resonance and the second term in the parentheses in Eq. (2) is strictly positive. It is convenient to introduce a dimensionless parameter γ, which characterizes the strength of interaction:

$$\gamma = \frac{mg}{\hbar^2\rho_0}, \tag{3}$$

where ρ_0 is the mean-field boson density.

In the absence of tunneling, the Hamiltonian [Eq. (1)] corresponds to the Lieb-Liniger model[34] of bosons with pointlike interactions. For this model, the effective Luttinger liquid description[35] agrees well with the exact solution for the ground-state properties[36] and the low-energy excitations (see Ref. 37 for the review). We thus expect that the "bosonization" procedure underlying the Luttinger liquid formalism also provides a good description of coupled condensates, at least when the value of the tunneling $t_\perp$ is not too large. The advantage of this low-energy description is that one can make explicit analytic calculations for both static and dynamic properties of the system. Physically, the main assumption justifying the use of this effective Hamiltonian is the absence of strong density fluctuations in the system. At sufficiently short distances (shorter than the healing length), this condition is violated. However, as one coarse grains the system, density fluctuations become weaker and the Luttinger liquid description becomes justified. Thus, the effective formalism can correctly describe phenomena at wavelengths longer than the healing length of the condensate.

Within the Luttinger liquid description, the Hamiltonian [Eq. (1)] splits into parts corresponding to individual condensates $H_{1,2}$ and the tunneling between them H_{tun}. The first two are characterized by the so-called Luttinger liquid parameter K and the sound velocity v_s, which we assume to be identical for both condensates:

$$H_{1,2} = \frac{v_s}{2}\int dx\left[\frac{\pi}{K}\Pi_{1,2}^2(x) + \frac{K}{\pi}(\partial_x\phi_{1,2})^2\right]. \tag{4}$$

Here, the phase field $\phi_{1,2}(x)$ is conjugate to the momentum $\Pi_{1,2}(x)$. The relation between the original bosonic operators $\Psi_j^\dagger$ and the new fields is given by the bosonization rule[35,37]

$$\Psi_j^\dagger(x) \sim \sqrt{\rho_0 + \Pi_j(x)}\sum_m e^{2im\Theta_j(x)}e^{-i\phi_j(x)}, \tag{5}$$

where $\partial\Theta_j(x)/\pi=\rho_0+\Pi_j(x)$.

In general, the relations of K and v_s to the original microscopic parameters m and g are not known analytically but can be deduced numerically from the exact solution of the Lieb-Liniger model. However, in the limit of large and small interactions, one can obtain approximate expressions.[37] Thus, for $\gamma \ll 1$,

$$v_s \approx v_F\frac{\sqrt{\gamma}}{\pi}\left(1 - \frac{\sqrt{\gamma}}{2\pi}\right)^{1/2}, \quad K \approx \frac{\pi}{\sqrt{\gamma}}\left(1 - \frac{\sqrt{\gamma}}{2\pi}\right)^{-1/2}, \tag{6}$$

and in the opposite limit $\gamma \gg 1$,

$$v_s \approx v_F\left(1 - \frac{4}{\gamma}\right), \quad K \approx 1 + \frac{4}{\gamma}. \tag{7}$$

Here, $v_F=\hbar\pi\rho_0/m$ is the Fermi velocity. Note that $v_sK=v_F$ as a consequence of the Galilean invariance. We also mention that the healing length ξ_h setting the scale of applicability of Eq. (4) is approximately equal to the interparticle distance, $\xi_h \sim 1/\rho_0$ at $\gamma \gg 1$, and becomes larger, $\xi_h \sim 1/\rho_0\sqrt{\gamma}$ at $\gamma \ll 1$. We note that experimentally, the coupling γ can be tuned through a very wide range[24] by either changing the density or transverse confinement length or by tuning $a_{3\mathrm{D}}$ near the Feshbach resonance [see Eq. (2)].

The tunneling part of the Hamiltonian H_{tun} can be described effectively as

$$H_{tun} = -2\Delta\cos(\phi_1 - \phi_2), \tag{8}$$

where we introduced the new parameter Δ. For small interactions $\gamma \lesssim 1$, we have $\Delta \approx t_\perp\rho_0$. At strong interactions, Δ will be renormalized and can significantly deviate from its bare form.

It is convenient to introduce the total and the relative phases $\phi_\pm=(\phi_1\pm\phi_2)/\sqrt{2}$ and $\Pi_\pm=(\Pi_1\pm\Pi_2)/\sqrt{2}$. Then, the Hamiltonian splits into symmetric and antisymmetric parts: $H=H_+ + H_-$, where

$$H_+ = \frac{v_s}{2}\int dx\left[\frac{\pi}{K}[\Pi_+(x)]^2 + \frac{K}{\pi}(\partial_x\phi_+)^2\right].$$

$$H_- = \frac{v_s}{2}\int dx\left[\frac{\pi}{K}[\Pi_-(x)]^2 + \frac{K}{\pi}(\partial_x\phi_-)^2\right] - 2\Delta\int dx\cos(\sqrt{2}\phi_-). \tag{9}$$

Note that the symmetric part of the Hamiltonian H_+ in our approximation is completely decoupled from the antisymmetric part H_-. However, higher nonlinear terms in Π and $\partial_x\phi$ will couple ϕ_+ and ϕ_-, so that the symmetric degrees of freedom can damp excitations of ϕ_-.

The Hamiltonian H_- can be simplified further. In particular, we can rescale units of time: $t \to tv_s$ to set $v_s=1$. It is also convenient to rescale the phase field ϕ and the conjugate momentum: $\phi_- \to \sqrt{\pi/K}\phi_-$ and $\Pi_- \to \sqrt{K/\pi}\Pi_-$. Then,

$$H_{-} = \frac{1}{2}\int dx[\Pi_{-}^{2}(x) + (\partial_x\phi_{-})^2 - 4\Delta\cos(\beta\phi_{-})], \quad (10)$$

where $\beta = \sqrt{2\pi/K}$. It is also convenient to introduce a parameter ξ,

$$\xi = \frac{\beta^2}{8\pi - \beta^2} = \frac{1}{4K-1}, \quad (11)$$

which we will use later in the text.

B. Quantum SG model: General facts

The spectrum of the quantum sine-Gordon Hamiltonian [Eq. (10)] depends on the value of β. For $4\pi < \beta^2 < 8\pi$, it consists of solitons and antisolitons. The point $\beta^2 = 4\pi$ corresponds to the free massive fermion theory. In this case, solitons and antisolitons correspond to particles and holes. For $0 < \beta^2 < 4\pi$ the spectrum, in addition to solitons and antisolitons, contains their bound states called *breathers*. Note that within the Lieb-Liniger model, $K \geqslant 1$, which implies that $\beta^2 \leqslant 2\pi$, so that we are always in the latter regime.[38] The number of breathers depends on the interaction parameter ξ and is equal to the integer part of $1/\xi$. We denote breathers by B_n for $n = 1, \ldots, [1/\xi]$. In the Gaussian limit of the sine-Gordon Hamiltonian [Eq. (10)], where one expands $\cos(\beta\phi_{-})$ to the quadratic order in ϕ_{-}, there is only one massive excitation corresponding to the lowest breather B_1. Solitons (kinks), antisolitons (antikinks), and breathers are massive particlelike excitations. Soliton and antisoliton masses in terms of the parameters of the Hamiltonian [Eq. (10)] were computed in Ref. 39 as follows:

$$M_s = \left(\frac{\pi\Gamma\left(\frac{1}{1+\xi}\right)\Delta}{\Gamma\left(\frac{\xi}{1+\xi}\right)}\right)^{(1+\xi)/2} \frac{2\Gamma\left(\frac{\xi}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1+\xi}{2}\right)}. \quad (12)$$

Breather masses are related to the soliton masses via

$$M_{B_n} = 2M_s \sin\left(\frac{\pi\xi n}{2}\right). \quad (13)$$

Note that at weak interactions (large K and small ξ), the lowest breather masses are approximately equidistant $M_{B_n} \propto n$, which suggests that these masses correspond to eigenenergies of a harmonic theory. There is indeed a direct analogy between the breathers in the sine-Gordon model and the energy levels of a simple Josephson junction.[32] In the Josephson junction, the energy levels also become approximately equidistant if the interaction (charging) energy is small.

Finally, at $\beta^2 = 8\pi$, the systems undergoes the Kosterlitz-Thouless transition so that for $\beta^2 > 8\pi$, the cosine term becomes irrelevant and system is described by the usual Luttinger liquid.

C. Possible experimental probes

We consider three different types of modulation experiments, which can reveal the spectrum of the quantum sine-Gordon model. We restrict ourselves to the linear response regime, which corresponds to small perturbations. Modulation of different parameters in the model focuses on different parts of quantum sine-Gordon (qSG) spectrum and provides complementary information. A schematic view of possible experimental setups is given in Fig. 1. The direction of axes of two parallel condensates is out of the plane of the figure.

In this paper, we discuss three types of periodic modulations: (a) modulation of the magnitude of the tunneling amplitude $\Delta(x,t)$ [this kind of modulation couples to the potential density operator $\cos(\beta\phi)$], (b) modulation of the relative potential difference between the two wells $V(x,t)$ [this modulation couples to the momentum operator $\Pi(x)$, which, in turn, is proportional to $\partial_t\phi$], and (c) modulation of the phase of the tunneling amplitude, e.g., $\Delta^*(x,t) = \Delta\exp(\pm ik(t)x)$. After redefinition of variables $\phi \to \tilde{\phi} = \phi \pm k(t)x/\beta$, the term corresponding to the density of the *topological current* operator $\partial_x\phi_{-}$ appears in the Hamiltonian. Note that the topological charge

$$Q = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} \partial_x\phi_{-} \quad (14)$$

is conserved and quantized as $\pm n$ with n being an integer (corresponding to the presence of solitons and antisolitons). We note that in linear response, the modulation of the type (c) is equivalent to modulating the relative current between the two condensates.

For each perturbation of the qSG model that we discussed, one can define an appropriate correlation function and susceptibility. According to the fluctuation-dissipation theorem, energy absorption (per unit time) during modulation experiments is given by the imaginary part of these susceptibilities. The latter are also known as structure factors. In Sec. IV, we present typical structure factors corresponding to experiments of type (a) (Fig. 2), type (b) (Fig. 3), and type (c) (Fig. 4). A general structure of the response function is given by a collection of peaks. These peaks can be classified into two groups: coherent, δ-function-like peaks coming from single breathers and the broader peaks coming from creating many (typically two)-particle excitations. From the brief overview given in Sec. II B, it is clear that the absorption spectrum significantly depends on the interaction parameter K. We give a detailed analysis of structure factors in Sec. IV.

III. QUANTUM SINE-GORDON MODEL AND FORM FACTORS OF ITS OPERATORS

The Hilbert space of the sine-Gordon model can be constructed from the asymptotic scattering states. The latter can be obtained by the action of operators $A_{a_k}(\theta)$ corresponding to elementary excitations on the vacuum state,

$$|\theta_1\theta_2\cdots\theta_n\rangle_{a_1,a_2,\ldots,a_n} = A_{a_1}^{\dagger}(\theta_1)A_{a_2}^{\dagger}(\theta_2)\cdots A_{a_n}^{\dagger}(\theta_n)|0\rangle, \quad (15)$$

where the operators $A_{a_k}(\theta_k)$ have internal index a_k corresponding to solitons ($a_k = +$), antisolitons ($a_k = -$), or breathers ($a_k=n$, $n=1,\ldots,[1/\xi]$) and depend on the rapidity θ_k which parametrizes the momentum and the energy of a single-quasiparticle excitation of mass M_{a_k} [see Eqs. (12) and (13)]: $E_a=M_a\cosh(\theta)$, $P_a=M_a\sinh(\theta)$. These states are complete and therefore satisfy the completeness relation:

$$1=\sum_{n=0}^{\infty}\sum_{\{a_i\}}\int\prod_{i=1}^{n}\frac{d\theta_i}{(2\pi)^n n!}|\theta_n\cdots\theta_1\rangle_{a_1\cdots a_n}^{a_1\cdots a_n}\langle\theta_1\cdots\theta_n|. \tag{16}$$

Elementary states or excitations in an integrable field theory can be described in terms of operators creating or annihilating asymptotic states. Their commutation relations involve the scattering matrix S,

$$A_a(\theta_1)A_b(\theta_2)=S_{ab}^{a'b'}(\theta_1-\theta_2)A_{b'}(\theta_2)A_{a'}(\theta_1),$$

$$A_a^\dagger(\theta_1)A_b^\dagger(\theta_2)=S_{ab}^{a'b'}(\theta_1-\theta_2)A_{b'}^\dagger(\theta_2)A_{a'}^\dagger(\theta_1),$$

$$A_a(\theta_1)A_b^\dagger(\theta_2)=2\pi\delta_{ab}\delta(\theta_1-\theta_2)+S_{ba'}^{b'a}(\theta_1-\theta_2)A_{b'}^\dagger(\theta_2)A_{a'}(\theta_1). \tag{17}$$

These relations are called the Zamolodchikov-Faddeev algebra. They generalize canonical commutation relations for bosons and fermions and reduce to them in some special cases. Since breathers are the bound states of solitons and antisolitons, the soliton-antisoliton ($a=$"+",$b=$"−") scattering matrix has corresponding imaginary poles at $\theta_n=i\pi(1-n\xi)$, $n=1,\ldots,[1/\xi]$.

Different components of S matrix are related to several allowed scattering processes (related to the configurations in the six-vertex model) and as a building block include the following quantity:

$$S_0(\theta)=-\exp\left[-i\int_0^\infty\frac{dx}{x}\frac{\sin\left(\frac{2x\theta}{\pi\xi}\right)\sinh\left(\frac{\xi-1}{\xi}x\right)}{\sinh(x)\cosh\left(\frac{x}{\xi}\right)}\right]. \tag{18}$$

The form factors $F^{\mathcal{O}}$ of a given operator $\mathcal{O}$ of an integrable model are the matrix elements in asymptotic states [Eq. (15)] created by elements A and $A^\dagger$ of the Zamolodchikov-Faddeev algebra [Eqs. (17)]. Explicitly,

$$F^{\mathcal{O}}(\theta_n\cdots\theta_1)_{a_n\cdots a_1}=\langle 0|\mathcal{O}(0,0)|\theta_n,\cdots\theta_1\rangle_{a_n\cdots a_1}. \tag{19}$$

Form factors (FF) satisfy a set of axioms and functional equations, which together allow us in principle, to determine their explicit forms. Using the so-called crossing relation and the translation invariance, all form factors can be expressed in terms of matrix elements of the form (19):

$$\langle 0|\mathcal{O}(x,t)|\theta_n,\cdots\theta_1\rangle_{a_n\cdots a_1}=\exp\left[i\sum_{j=1}^{n}(E_jt+P_jx)\right]\times\langle 0|\mathcal{O}(0,0)|\theta_n,\cdots\theta_1\rangle_{a_n\cdots a_1}, \tag{20}$$

where $E_j=M_{a_j}\cosh(\theta_j)$, $P_j=M_{a_j}\sinh(\theta_j)$. Note that in the noninteracting limit of some generic model, Zamolodchikov-Faddev algebra reduces either to Bose or Fermi canonical commutation relations. In this case, form factors simply reduce to the coefficients of the expansion of the operator $\mathcal{O}$ in the second quantized form.

Explicit analytical expressions for the form factors depend on the specific type of the operator $\mathcal{O}$. In this paper, we will be interested in two particular types of operators: (i) the current operator in the sine-Gordon model:

$$J^\mu=-\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi_-, \tag{21}$$

where $\mu=0,1$ correspond to time t and space x components, and (ii) the other operator corresponding to the trace of the stress-energy, tensor $T^{\mu\nu}=\partial_\mu\partial_\nu\phi-\delta_{\mu\nu}\mathcal{L}$, where $\mathcal{L}$ is the Lagrangian,

$$T\equiv\mathrm{Tr}(T_\nu^\mu)=4\Delta\cos(\beta\phi_-). \tag{22}$$

These two operators have different properties with respect to the Lorentz group and different charge parity. Therefore, their properties can be distinguished by the topological charge [Eq. (14)] and the charge conjugation operator C defined by

$$C|0\rangle=|0\rangle,\quad CA_s^\dagger(\theta)C^{-1}=A_{\bar{s}}^\dagger(\theta), \tag{23}$$

$$CA_{B_n}^\dagger(\theta)C^{-1}=(-1)^nA_{B_n}^\dagger(\theta). \tag{24}$$

The action of both the current operator and $\cos(\beta\phi_-)$ do not change the value of Q.

There is extensive literature on computation of form factors for the quantum sine-Gordon model. To our knowledge, such analysis was initiated in Ref. 40 and extended later in Ref. 41. These developments are summarized in the book [42] and more recently in Refs. 43–45.

If there are bound states in the theory, like breathers in our case, the form factors which include these bound states can be constructed using the residue at the poles of the soliton-antisoliton form factors. To get form factors corresponding to higher breathers, one can use a so-called fusion procedure. The receipt is that the breather B_{n+m} appears as a bound state of the B_n and B_m breathers, $B_n+B_m\to B_{n+m}$. Thus, the residue of the pole in the form factor $F_{B_nB_m}$ will correspond to the form factor $F_{B_{n+m}}$. For example, to construct the form factor which includes the second breather B_2, we can use the form factor with $n+2$ breathers B_1 and the fusion formula

$$i\,\mathrm{Res}_{\epsilon\to 0}\,F_{a_1\cdots a_n 11}\left(\theta_1,\ldots,\theta_n,\theta_{n+1}+iU_{12}^1-\frac{\epsilon}{2},\theta_{n+1}-iU_{12}^1+\frac{\epsilon}{2}\right)=\Gamma_{11}^2F_{a_1\cdots a_n 2}(\theta_1,\ldots,\theta_n,\theta_{n+1}),\tag{25}$$

where the fusion angle $U_{12}^1=\pi\xi/2$ and the three-particle coupling $\Gamma_{11}^2=\sqrt{2}\tan(\pi\xi)$ is given by the residue of the S matrix [Eq. (18)].

A. Soliton-antisoliton form factors

1. $\mathcal{O}=e^{i\beta\phi}$

For the sine-Gordon model, different form factors for the phase operators $e^{i\beta\phi}$ have been found in Refs. 42 and 43. In particular, the nonvanishing "$+-$" (soliton-antisoliton) form factor is given by

$$\langle 0|e^{i\beta\phi(0,0)}|A_\pm^\dagger(\theta_2)A_\mp^\dagger(\theta_1)\rangle=\mathcal{G}_\beta\exp[I(\theta_{12})]\times\frac{2i\cot\left(\frac{\pi\xi}{2}\right)\sinh(\theta_{12})}{\xi\sinh\left(\frac{\theta_{12}+i\pi}{\xi}\right)}\times\exp\left(\mp\frac{\theta_{12}+i\pi}{2\xi}\right),\tag{26}$$

where $\theta_{12}=\theta_1-\theta_2$, and

$$I(\theta)=\int_0^\infty\frac{dt}{t}\frac{\sinh^2\left[t\left(1-\frac{i\theta}{\pi}\right)\right]\sinh[t(\xi-1)]}{\sinh[2t]\cosh[t]\sinh[t\xi]}.\tag{27}$$

Equation (26), as well as the normalization of general $\cos(\beta\phi_-)$-type form factors, includes the following vacuum-vacuum amplitude[46]:

$$\mathcal{G}_\beta=\langle 0|e^{i\beta\phi}|0\rangle=\Delta^\xi\frac{(1+\xi)\tan\left(\frac{\pi\xi}{2}\right)\Gamma^2\left(\frac{\xi}{2}\right)}{2\pi\Gamma^2\left(\frac{1+\xi}{2}\right)}\left[\frac{\pi\Gamma\left(\frac{1}{1+\xi}\right)}{\Gamma\left(\frac{\xi}{1+\xi}\right)}\right]^{1+\xi}.\tag{28}$$

Higher-order soliton-antisoliton form factors (e.g., with two solitons and two antisolitons) are also available in the literature.[42] For generic values of K, they are given by multiple integrals, whereas for the half-integer values of $4K$ (in our notations), the expressions considerably simplify. In the next section, we explain that the relative contribution to the operator's expectation value from the states which include many soliton-antisoliton pairs is very small and can be neglected.

2. $\mathcal{O}=\partial_\mu\phi_-$

The form factors for these operators can be deduced from the known form factors of the current operator J^μ. In particular, the form factor for the operator $\partial_t\phi_-$ is

$$F_{+,-}^{\partial_t\phi_-}(\theta_1,\theta_2)=-F_{-,+}^{\partial_t\phi_-}(\theta_1,\theta_2)=\frac{4\pi^{3/2}\sqrt{2}M_s}{\beta}\frac{\cosh\left(\frac{\theta_1+\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_2}{2}\right)}{\cosh\left(\frac{\theta_1-\theta_2+i\pi}{2\xi}\right)}I(\theta_{12}).\tag{29}$$

The FF for the $\partial_x\phi_-$ is given by

$$F_{+,-}^{\partial_t\phi_-}(\theta_1,\theta_2)=-F_{-,+}^{\partial_t\phi_-}(\theta_1,\theta_2)=\frac{4\pi^{3/2}\sqrt{2}M_s}{\beta}\frac{\sinh\left(\frac{\theta_1+\theta_2}{2}\right)\sinh\left(\frac{\theta_1-\theta_2}{2}\right)}{\cosh\left(\frac{\theta_1-\theta_2+i\pi}{2\xi}\right)}I(\theta_{12}).\tag{30}$$

Multi-soliton-antisoliton form factors of this operator can be found in Ref. 42 in terms of integrals. Their relative contribution is small as well[29] and we will neglect them.

B. Breather form factors

1. $\mathcal{O}=e^{i\beta\phi}$

The one-breather form factors of the type $\langle 0|e^{i\beta\phi}|B_n(0)\rangle\equiv F_{B_n}$ can be computed from the residue of the soliton-antisoliton form factors at points $\theta_n=i\pi(1-n\xi)$, or, equivalently, from the procedure of fusion of several breathers. For example, the breather B_2 is the bound state of two breathers B_1. This procedure is known as a bootstrap approach,

$$F_{B_n}^{\exp(i\beta\phi)}=\frac{\mathcal{G}_\beta\sqrt{2}\cot\left(\frac{\pi\xi}{2}\right)\sin(\pi n\xi)\exp[I(-\theta_n)]e^{i\pi n/2}}{\sqrt{\cot\left(\frac{\pi\xi n}{2}\right)\prod_{s=1}^{n-1}\cot^2\left(\frac{\pi\xi s}{2}\right)}}.\tag{31}$$

We note that Eq. (31) reveals the parity property of breathers: odd breathers are antisymmetric with respect to the charge symmetry transformation, whereas even breathers are symmetric.

The form factors of several breathers can be derived as well. Since the breather B_1 is like a fundamental particle in a theory, all higher-level breathers form factors can be expressed via a fusion procedure of B_1 form factors. Because of the parity, the nonzero form-factors for the operator $\cos(\beta\phi)$ contain either even breathers or even number of odd breathers. It is thus important to have an explicit expression for the $B_1\cdots B_1$ form factors

$$F_{B_1\cdots B_1}^{\cos(\beta\phi)}=\langle 0|\cos(\beta\phi)|B_1(\theta_{2n})\cdots B_1(\theta_1)\rangle=\frac{1}{2}\mathcal{G}_\beta\lambda^{2n}\prod_{1\le i<j\le 2n}\mathcal{R}(\theta_k-\theta_j)\frac{\det(\Sigma'_{ij})}{\det(\Sigma_{ij})}.\tag{32}$$

Here, the $(2n-1)\times(2n-1)$ matrices

$$\Sigma_{ij}=\sigma_{2i-j}, \quad \Sigma'_{ij}=\sigma_{2i-j}\frac{\sin[\pi\xi(i-j+1)]}{\pi\xi}$$

are expressed in terms of symmetric polynomials

$$\sigma_k \equiv \sigma_k^{(2n-1)}=\sum_{i_1<i_2<\cdots<i_k}^{2n-1} x_{i_1}\cdots x_{i_k},$$

with $x_i=e^{\theta_i}$,

$$\lambda = 2\cos\left[\frac{\pi\xi}{2}\right]\sqrt{2\sin\left[\frac{\pi\xi}{2}\right]}\exp\left[-\int_0^{\pi\xi}\frac{dt}{2\pi}\frac{t}{\sin t}\right],$$

$$\mathcal{R}(\theta)=N\exp\left[8\int_0^{\infty}\frac{dt}{t}\frac{\sinh(t)\sinh(t\xi)\sinh(t(1+\xi))}{\sinh^2(2t)}\times\sinh^2\left(1-i\frac{\theta}{\pi}\right)\right],$$

$$N=\exp\left[4\int_0^{\infty}\frac{dt}{t}\frac{\sinh(t)\sinh(t\xi)\sinh(t(1+\xi))}{\sinh^2(2t)}\right]. \quad (33)$$

Here, the function $\mathcal{R}(\theta)$ satisfies a useful relation which allows us to evaluate its residue at $\theta=-i\pi(1-n\xi)$,

$$\mathcal{R}(\theta)\mathcal{R}(\theta\pm i\pi)=\frac{\sinh(\theta)}{\sinh(\theta)\mp i\sinh(\pi\xi)}. \quad (34)$$

The formula [Eq. (32)] can be shown to agree both with the sine-Gordon bosonization by Lukyanov[44] and the Bethe-ansatz-based method of Ref. 45.

The form factor for the B_2-B_2 breathers can be derived from Eq. (32) using the fusion procedure:

$$F_{B_2B_2}^{\cos(\phi)}(\theta)=\frac{\mathcal{G}_\beta\lambda^4}{2}\tan(\pi\xi)\left(1+\frac{1}{\cosh(\theta)+\cos(\pi\xi)}\right)\times\frac{\mathcal{R}^2(\theta)\mathcal{R}(-\theta-i\pi\xi)\mathcal{R}(-\theta+i\pi\xi)}{\mathcal{R}(-i\pi(1+\xi))}.$$

2. $\mathcal{O}=\partial_t\phi_-$

Because of the charge reflection symmetry, odd-type one-breather form factors exist for this operator. They can be found either directly from the poles of the soliton-antisoliton form factor [Eq. (29)] or via the relation $\langle 0|\partial_t\phi_-|B_n(\theta)\rangle = \lim_{a\to 0}1/iaM_{B_n}\cosh(\theta)\langle 0|e^{ia\phi_-}|B_n(\theta)\rangle$. The result is

$$F_{B_m}^{\partial_t\phi}(\theta)=\left[\cot\left(\frac{\pi m\xi}{2}\right)\prod_{s=1}^{m-1}\tan^2\left(\frac{\pi s\xi}{2}\right)\right]^{1/2}\times\frac{M_{B_m}\sqrt{2}(2\pi)^{3/2}\xi}{\beta}\exp(I(-\theta_m))\cosh(\theta), \quad (35)$$

$$F_{B_m}^{\partial_x\phi}(\theta)=\left[\cot\left(\frac{\pi m\xi}{2}\right)\prod_{s=1}^{m-1}\tan^2\left(\frac{\pi s\xi}{2}\right)\right]^{1/2}\times\frac{M_{B_m}\sqrt{2}(2\pi)^{3/2}\xi}{\beta}\exp(I(-\theta_m))\sinh(\theta), \quad (36)$$

where m is odd.

To compute form factors for several breathers, we note that there is a correspondence between the lowest B_1 breather sector of the sine-Gordon theory and the sinh-Gordon theory. Using this correspondence the B_1 form factors can be obtained via the recursion formula[47] from the form factors of φ_{sinh}—operator in the sinh-Gordon model. This recursion gives us odd-number nonzero B_1 form factors

$$F_{B_1^{(1)}\ldots B_1^{(2n+1)}}^{\partial_t\phi_-}=\frac{(2\pi)^{3/2}\lambda^{2n+1}\xi}{\beta\sin(\pi\xi)}M_{B_1}\left[\sum_{n=1}^{2n+1}\cosh(\theta_n)\right]\times\sigma_{2n+1}^{(2n+1)}P_{2n+1}(x_1,\ldots,x_{2n+1})\prod_{i<j}\frac{\mathcal{R}(\theta_i-\theta_j)}{x_i+x_j}, \quad (37)$$

where $x_i=e^{\theta_i}$ and first polynomials P are given by[47]

$$P_3(x_1,x_2,x_3)=1, \quad (38)$$

$$P_5(x_1,\ldots,x_5)=\sigma_2\sigma_3-4\cos^2(\pi\xi)\sigma_5. \quad (39)$$

These formulas allow us to compute, for example, the B_1-B_2 form factor, which is the first nontrivial two-particle form factor after the soliton-antisoliton form factor:

$$F_{B_1B_2}^{\partial_x\phi}(\theta_1,\theta_2)=(M_{B_1}\sinh(\theta_1)+M_{B_2}\sinh(\theta_2))\times\tilde{F}_{B_1B_2}^{\partial_x\phi}(\theta_{12}),$$

$$\tilde{F}_{B_1B_2}^{\partial_x\phi}(\theta_{12})=\frac{\pi^{3/2}\lambda^3\xi}{\sqrt{8}\sin(\pi\xi)\beta}\times\frac{\sqrt{\tan(\pi\xi)}\mathcal{R}\left(\theta_{23}+\frac{i\pi\xi}{2}\right)\mathcal{R}\left(\theta_{23}-\frac{i\pi\xi}{2}\right)}{\cos\left(\frac{\pi\xi}{2}\right)\left(\cosh(\theta)+\cos\left(\frac{\pi\xi}{2}\right)\right)\mathcal{R}(-i\pi(1+\xi))}.$$

It is remarkable that the further fusion of B_1 and B_2 particles produces the B_3 form-factor, which is identical to the result from Eq. (35). This agreement is a direct manifestation of the *bootstrap*.[14]

IV. LINEAR RESPONSE THEORY FOR COUPLED CONDENSATES

Experimentally, one can modulate the relative coupling between two condensates by applying small modulations in space and/or time. Moreover, one can change the densities in both condensates such that the relative density can vary. Following the previous discussion, we will consider three relevant operators which describe different physical perturbations

$$(a)\mathcal{O}_T(x,t) = \cos(\beta\phi_-), \tag{40}$$

$$(b)\mathcal{O}_{J_1}(x,t) = \partial_t\phi_-, \tag{41}$$

$$(c)\mathcal{O}_{J_0}(x,t) = \partial_x\phi_-. \tag{42}$$

These operators directly correspond to the experimental setups (a), (b), and (c) discussed in the Introduction. The conserved topological charge in these notations is given by the integral of J^0.

In the linear response regime we are interested in computing two-point correlation functions

$$\langle\mathcal{O}(x,t)\mathcal{O}^\dagger(0,0)\rangle, \tag{43}$$

where the expectation value is taken over the ground state of the unperturbed system. Inserting the completeness relation [Eq. (16)] and using the relativistic invariance allow us to express this average in the following weighted sum over the intermediate states:

$$\langle\mathcal{O}(x,t)\mathcal{O}^\dagger(0,0)\rangle = \sum_{n=0}^{\infty}\sum_{\{a_i\}}\int\prod_{i=1}^{n}\frac{d\theta_i}{(2\pi)^n n!}\exp\left(\sum_{j=1}^{n}P_jx - E_jt\right) \times|\langle 0|\mathcal{O}(0,0)|\theta_n\cdots\theta_1\rangle_{a_n\cdots a_1}|^2. \tag{44}$$

It is convenient to define the dynamical structure factor as a Fourier transform of this quantity,

$$S^{\mathcal{O}}(q,\omega) = \mathrm{Im}\left[\int\int_{-\infty}^{\infty}dxdte^{i\omega t-iqx}\langle\mathcal{O}(x,t)\mathcal{O}^\dagger(0,0)\rangle\right] = 2\pi\sum_{n=0}^{\infty}\sum_{a_i}\int\prod_{i=1}^{n}\frac{d\theta_i}{(2\pi)^n n!}|F^{\mathcal{O}}(\theta_1\cdots\theta_n)_{a_1\cdots a_n}|^2 \times\delta\left(q-\sum_j M_{a_j}\sinh\theta_j\right)\delta\left(\omega-\sum_j M_{a_j}\cosh\theta_j\right), \tag{45}$$

where $F^{\mathcal{O}}$ is the corresponding form factor of the operator $\mathcal{O}$. We comment here on the structure of expression (45). First, it is represented as a sum of contributions coming from different excited states of the Hamiltonian. Second, it is known from rather general phase-space arguments[48,49] that the sum over intermediate states of form (45) converges rapidly. In particular, it was shown in Ref. 29 that the two-soliton–two-antisoliton contribution is somewhat 500 smaller than the soliton-antisoliton one. Therefore, in general, the contributions with small number of excited particles will dominate the result. The expression for the structure factor [Eq. (45)] can be also rewritten in the following sum:

$$S^{\mathcal{O}}(q,\omega) = S^{\mathcal{O}}_{(1)}(q,\omega) + \frac{1}{4\pi}S^{\mathcal{O}}_{(2)}(q,\omega) + \frac{1}{24\pi^2}S^{\mathcal{O}}_{(3)}(q,\omega) + \cdots. \tag{46}$$

The specific form of $S^{\mathcal{O}}_{(n)}(q,\omega)$ significantly depends on the charge parity of the operator and the topological charge parity. The current operator is odd with respect to the C parity, whereas $\cos(\beta\phi_-)$ is even. All operators we consider do not change the topological charge.

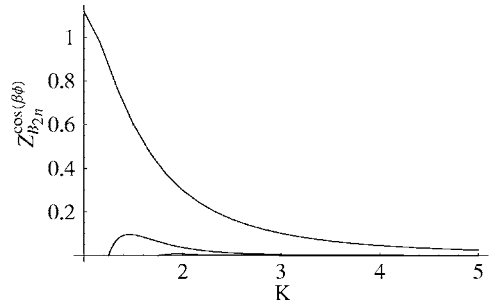

FIG. 6. Spectral weights $|F^{\cos(\phi)}_{B_n}|^2$ of single-breather contributions corresponding to the scheme (a). Shown here are contributions for (from top to bottom) B_2, B_4, and B_6 (very small).

Explicit expressions for the $S^{\cos(\beta\phi)}(q,\omega)$ have already appeared in literature[30,31] in the context of one-dimensional spin-1/2 systems discussed in the Introduction. We present them here for completeness.

(a) In this case, $\mathcal{O}=\cos(\beta\phi_-)$. Then, the single particle contribution to the structure factor is given by

$$S^{\cos(\beta\phi)}_{(1)}(q,\omega) = \sum_{n=1}^{[1/\xi]}\mathcal{Z}^{\cos(\beta\phi)}_{B_{2n}}\delta(s^2 - M^2_{B_{2n}}), \tag{47}$$

where $s^2=\omega^2-q^2$. Thus, $S_{(1)}$ corresponds to excitation of isolated even breathers. Here, the spectral weights are

$$\mathcal{Z}^{\cos(\beta\phi)}_{B_{2n}} = |F^{\cos(\beta\phi)}_{B_{2n}}|^2, \tag{48}$$

where $F^{\cos(\beta\phi)}_{B_{2n}}$ is given in Eq. (31). So, this contribution corresponds to the absorption peaks at $\omega=\omega_{B_{2n}}$ (for $q=0$). The spectral weights are illustrated in Fig. 6 for $n=1,2,3$. Note that these weights decrease with increasing n as well as with increasing K. This plot suggests that breathers with small n dominate the absorption.

The two-particle contribution to the structure factor corresponding to excitation of particles A_1 and A_2 with masses M_{A_1} and M_{A_2} can be generally expressed as

$$S^{\mathcal{O}}_{(2)}(q,\omega) = \mathrm{Re}\left[\frac{|F_{A_1A_2}(\theta_{12})|^2}{\sqrt{(s^2-M^2_{A_1}-M^2_{A_2})^2-4M^2_{A_1}M^2_{A_2}}}\right], \tag{49}$$

where

$$\theta_{12} = \theta(q,\omega) = \mathrm{arccosh}\left(\frac{s^2-M^2_{A_1}-M^2_{A_2}}{2M_{A_1}M_{A_2}}\right). \tag{50}$$

Direct substitution of the single breather form factor [Eq. (31)], the soliton-antisoliton form factor [Eq. (26)], and breather-breather form factors [Eqs. (32) and (35)] from the previous section gives the results for the corresponding structure factors. Note that the soliton-antisoliton contribution has to be weighted by a factor of 2. The results are shown in Figs. 2 and 3 for some fixed values of the Luttinger parameter K and the interchain coupling Δ. We see that the generic form of the structure factor (and this of the absorp-

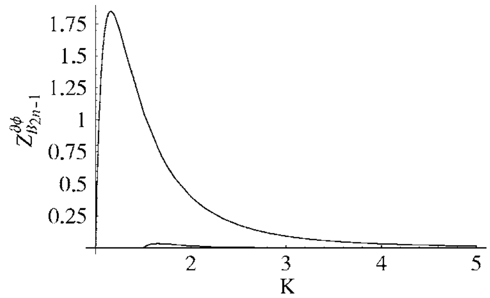

FIG. 7. Spectral weights $|F^{\partial_t\phi}_{B_n}|/E_{B_n}$ ($|F^{\partial_t\phi}_{B_n}|/P_{B_n}$) of single-breather contributions appearing in setup (b) and (c). Shown here are contributions for (from top to bottom) B_1, B_3, and B_5.

tion peaks) has a coherent part represented by a sequence of δ peaks corresponding to coherent isolated even breather contributions and the incoherent part corresponding to different two-particle contributions. In Figs. 2 and 3, we show the contributions from B_1B_1, A_+A_-, B_2B_2 (incoherent part), and B_2, B_4 (coherent peaks) form factors.

(b) For $\mathcal{O}=\partial_t\phi_-$, the general form of the structure factor is different:

$$S^{\partial_t\phi}_{(1)}(q,\omega)=\mathcal{Z}^{\partial_t\phi}_{B_{2n-1}}\omega^2\delta(s^2-M^2_{B_{2n-1}}), \tag{51}$$

where $\mathcal{Z}^{\partial_t\phi}_{B_{2n-1}}=|F^{\partial_t\phi}_{B_{2n-1}}/E_{B_{2n-1}}|^2$ is given by Eq. (35). The spectral weights $\mathcal{Z}^{\partial_t\phi}_{B_{2n-1}}$ are compared in Fig. 7 for $n=1,2,3$.

Similarly, the soliton-antisoliton contributions are

$$S^{\partial_t\phi}_{(2),s\bar{s}}(q,\omega)=\mathrm{Re}\left[\frac{\omega^2\sqrt{s^2-4M_s^2}}{s^3}\frac{|I(\theta_{12})|^2}{\cosh\left(\frac{\theta_{12}}{\xi}\right)+\cos\left(\frac{\pi}{\xi}\right)}\right], \tag{52}$$

where θ_{12} is defined in Eq. (50) with $M_{A_1}=M_{A_2}=M_s$ and $I(\theta)$ is given by Eq. (27). Finally, the two breather contributions are

$$S^{\partial_t\phi}_{(2),B_1B_2}(q,\omega)=\mathrm{Re}\left[\frac{\omega^2|\tilde{F}(\theta_{12})|^2}{\sqrt{(s^2-M^2_{B_1}-M^2_{B_2})^2-(2M_{B_1}M_{B_2})^2}}\right], \tag{53}$$

where θ_{12} is defined in Eq. (50) and $\tilde{F}(\theta_{12})$ is introduced in Eq. (40). In Fig. 4, we show the structure factor for the operator $\mathcal{O}=\partial_t\phi_-$ for $K=1.6$. As before, we include contributions corresponding to excitations of B_1, B_3, A_+A_-, and B_1B_2.

(c) The structure factor for this type of modulation is very much the same as for case (b). In particular, the ω dependence in the nominator of Eqs. (51)–(53) should be replaced by q.

V. SUMMARY AND OUTLOOK

We considered a system of two coupled one-dimensional condensates. We showed that the low-energy dynamics of this system can be described by the quantum sine-Gordon model (Sec. II A). This model is integrable and supports collective excitations (solitons, antisolitons, and breathers). To reveal these excitations, we propose to study modulations of the tunneling amplitude [type (a)], of the population imbalance [type (b)], and of the tunneling phase [type (c)]. Corresponding experiments provide complementary information about the structure of excitations of the quantum sine-Gordon model. The modulations of type (a) reveal *even-number* coherent breather peaks and even two-breather contributions, whereas modulations of type (b) and type (c) couple to the odd sector of the spectrum, so that the coherent peaks correspond to *odd* breathers and odd two-particle excitations. All types of modulations show the soliton-antisoliton response. The effect of the Luttinger interaction parameter K of the individual condensates is twofold: (i) the spectrum content is entirely determined by the strength of K, as it is explained in Sec. II B, and (ii) the spectral weights of coherent single-particle excitations B_n with $n\geqslant 2$ decrease significantly with increasing K. This result is not surprising since at large K, we anticipate that the lowest-energy excitations are well described by the Gaussian model of the massive scalar field. In this limit, there are only massive phonon-like excitations corresponding to B_1. For the type (a) modulation, these excitations can be created only in pairs giving raise to the continuum contribution to the spectral function, while for type (b) and type (c) modulations, the lowest-energy contribution comes from excitations of isolated B_1 breathers and then there is a three-particle continuum. We note that in the weakly interacting limit $K\gg 1$, the isolated B_1 peak corresponds to exciting Josephson oscillations between the two condensates. Also, we note that the soliton-antisoliton contribution to the spectral function rapidly diminishes with increasing K. The experimental visibility of various contribution will also be determined by the tunneling strength Δ. In general, one can observe an approximate scaling of structure factors with increasing Δ.

We comment that the idea behind modulation experiments is similar to exciting parametric resonance in a usual harmonic oscillator. This analogy becomes even more transparent for the special case of zero-dimensional condensates, which is equivalent to the Josephson junction. At weak nonlinearity the Josephson junction in turn is equivalent to a harmonic oscillator. The type (a) modulation of the tunneling amplitude is analogous to the modulation of the mass of this oscillator and the type (b) modulation is similar to the modulation of the equilibrium position of this oscillator. The strongest parametric excitation of a harmonic oscillator with frequency ω_0 occurs at $\omega=2\omega_0$ for the modulation of type (a) and at $\omega=\omega_0$ for the modulation of the type (b). These transitions, in turn, correspond to excitations of the second (a) and the first (b) energy levels in the Josephson junction. As we discussed in Ref. 32, the first energy level of the Josephson junction corresponds to the B_1 breather in the sine-Gordon model, the second energy level does to the B_2 breather, and so on. Indeed, as we showed above, the contributions from isolated B_2 and B_1 breathers to the absorption are dominant for the type (a) and type (b) modulations, respectively. We can also come to similar conclusions using the classical analysis[50,51] of the perturbed SG model for small ξ limit. One needs to bear in mind that here we deal with a nonlinear system and the parametric resonance strictly applies to a harmonic oscillator. So the response of the system to modulations is more complicated. However, qualitatively the picture remains very similar in this limit.

Experimentally the absorption can be enhanced by increasing the magnitude of the modulation signal. Even though our analysis is limited only to weak perturbations, where one can use the linear response, we expect that the overall picture will not significantly change in the nonlinear regime. However, one always needs to make sure that the nonlinear effects do not cause various instabilities in the system. For example, the modulation of type (c) is limited by the possible commensurate-incommensurate phase transition,[5] which occurs if the breather's gap gets comparable with the change of the chemical potential per particle. Another effect which exists on the classical level is a dissociation of breathers into decoupled soliton-antisoliton pair for relatively strong perturbations.[52]

It is important to realize that since we deal with an integrable or nearly integrable system, the energy absorption is not necessarily related to the loss of the phase coherence, which is usually measured in standard time of flight experiments. Indeed, the two quantities are related if the absorbed energy is quickly redistributed among all possible degrees of freedom. However, in integrable or nearly integrable model, this is not the case. For example, in a recent experiment by Kinoshita *et al.*,[26] it was demonstrated that there is no thermalization in a single one-dimensional Bose gas even after extremely long waiting times. Thus one possibility to measure the absorption is to slowly drive the excited system into the nonintegrable regime where the thermalization occurs and then perform conventional measurements. The other possibility is to directly measure the susceptibilities. For example, for the type (a) modulation, one can measure the expectation value of the phase difference between the condensates. For the type (b) modulation, one can measure the population imbalance between the two condensates, and for the type (c) modulation, one can measure the relative current between the two condensates. In principle, one can always measure the loss of the phase contrast as it was done in Ref. 53, however, the sensitivity of such probe in integrable systems can be very small.

ACKNOWLEDGMENTS

We would like to thank I. Affleck, E. Altman, I. Bloch, M. Lukin, S. Lukyanov, J. Schmiedmayer, and V. Vuletic for useful comments and discussions. This work was partially supported by the NSF, Harvard-MIT CUA, and AFOSR. V.G. was also supported by the Swiss National Science Foundation. A.P. was supported by AFOSR YIP.

1. S. Coleman, Phys. Rev. D **11**, 2088 (1975).
2. R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975); **12**, 1840 (1975); **12**, 2443 (1975).
3. D. R. Nelson and B. I. Halperin, Phys. Rev. B **19**, 2457 (1979); J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, *ibid.* **16**, 1217 (1977); V. Ambegaokar, B. I. Halperin, D. R. Nelson, and E. D. Siggia, *ibid.* **21**, 1806 (1980).
4. M. P. A. Fisher and G. Grinstein, Phys. Rev. Lett. **60**, 208 (1988).
5. G. I. Japaridze and A. A. Nersesyan, JETP Lett. **27**, 334 (1978); V. L. Pokrovsky and A. L. Talapov, Phys. Rev. Lett. **42**, 65 (1979); H. J. Schulz, Phys. Rev. B **22**, 5274 (1980).
6. M. Kardar, Phys. Rev. B **33**, 3125 (1986).
7. M. P. A. Fisher, P. B. Weichman, G. Grinstein, and D. S. Fisher, Phys. Rev. B **40**, 546 (1989).
8. A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, 2004).
9. S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A **9**, 3841 (1994); S. Ghoshal, *ibid.* **9**, 4801 (1994).
10. H. Saleur, arXiv:cond-mat/9812110 (unpublished); arXiv:cond-mat/0007309 (unpublished).
11. P. Fendley, F. Lesage, and H. Saleur, J. Stat. Phys. **85**, 211 (1996).
12. Z. Gulácsi and M. Gulácsi, Adv. Phys. **47**, 1 (1998).
13. L. D. Faddeev and L. A. Takhtajan, Teor. Mat. Fiz. **21**, 160 (1974); L. D. Faddeev, Zh. Eksp. Teor. Fiz. Pis'ma Red. **21**, 141 (1975) [JETP Lett. **21**, 64 (1975)]; V. E. Korepin and L. D. Faddeev, Teor. Mat. Fiz. **25**, 1039 (1975); V. E. Korepin, P. P. Kulish, and L. D. Faddeev, Zh. Eksp. Teor. Fiz. Pis'ma Red. **21**, 302 (1975) [JETP Lett. **21**, 138 (1975)].
14. Al. Zamolodchikov and A. Zamolodchikov, Ann. Phys. (N.Y.) **120**, 253 (1979).
15. M. Oshikawa and I. Affleck, Phys. Rev. Lett. **79**, 2883 (1997); I. Affleck and M. Oshikawa, Phys. Rev. B **60**, 1038 (1999); **62**, 9200 (2000).
16. F. H. L. Essler, A. Furusaki, and T. Hikihara, Phys. Rev. B **68**, 064410 (2003).
17. S. A. Zvyagin, A. K. Kolezhuk, J. Krzystek, and R. Feyerherm, Phys. Rev. Lett. **93**, 027201 (2004).
18. D. C. Dender, P. R. Hammar, D. H. Reich, C. Broholm, and G. Aeppli, Phys. Rev. Lett. **79**, 1750 (1997).
19. T. Asano, H. Nojiri, Y. Inagaki, J. P. Boucher, T. Sakon, Y. Ajiro, and M. Motokawa, Phys. Rev. Lett. **84**, 5880 (2000).
20. M. Kenzelmann, Y. Chen, C. Broholm, D. H. Reich, and Y. Qiu, Phys. Rev. Lett. **93**, 017204 (2004).
21. S. Hofferberth, I. Lesanovsky, B. Fischer, J. Verdu, and J. Schmiedmayer, Nat. Phys. **2**, 710 (2006).
22. T. Schumm, S. Hofferberth, L. M. Andersson, S. Wildermuth, S. Groth, I. Bar-Joseph, J. Schmiedmayer, and P. Krüger, Nat. Phys. **1**, 57 (2005).
23. G.-B. Jo, Y. Shin, S. Will, T. A. Pasquini, M. Saba, W. Ketterle, D. E. Pritchard, M. Vengalattore, and M. Prentiss, Phys. Rev. Lett. **98**, 030407 (2007).
24. T. Kinoshita, T. Wenger, and D. Weiss, Science **305**, 1125 (2004).
25. T. Kinoshita, T. Wenger, and D. S. Weiss, Phys. Rev. Lett. **95**, 190406 (2005).
26. T. Kinoshita, T. Wenger, and D. S. Weiss, Nature (London) **440**, 900 (2006).

[27] J. Sebby-Strabley, M. Anderlini, P. S. Jessen, and J. V. Porto, arXiv:cond-mat/0602103 (unpublished).
[28] F. H. L. Essler, F. Gebhard, and E. Jeckelmann, Phys. Rev. B **64**, 125119 (2001).
[29] D. Controzzi, F. H. L. Essler, and A. M. Tsvelik, arXiv:cond-mat/0011439 (unpublished); D. Controzzi, F. H. L. Essler, and A. M. Tsvelik, Phys. Rev. Lett. **86**, 680 (2001).
[30] F. H. L. Essler and A. M. Tsvelik, Phys. Rev. B **57**, 10592 (1998); F. H. L. Essler, A. M. Tsvelik, and G. Delfino, *ibid.* **56**, 11001 (1997); A. Iucci, M. A. Cazalilla, A. Ho, and T. Giamarchi, Phys. Rev. A **73**, 041608(R) (2006).
[31] F. H. L. Essler and R. M. Konik, in *I. Kogan Memorial Collection*, edited by M. Shifman, A. Vainshtein, and J. Wheater (World Scientific, Singapore, 2005), p. 684.
[32] V. Gritsev, E. Demler, M. Lukin, and A. Polkovnikov, arXiv:cond-mat/0702343 (unpublished).
[33] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998).
[34] E. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963); E. Lieb, *ibid.* **130**, 1616 (1963).
[35] F. D. M. Haldane, Phys. Rev. Lett. **47**, 1840 (1981).
[36] J.-S. Caux, P. Calabrese, and N. A. Slavnov, J. Stat. Mech.: Theory Exp. (2007) P01008; J.-S. Caux and P. Calabrese, Phys. Rev. A **74**, 031605(R) (2006).
[37] M. A. Cazalilla, J. Phys. B **37**, S1 (2004).
[38] We note that the Luttinger parameter K in coupled condensates can be additionally renormalized by the tunneling term and deviate from the single-condensate value. However, we anticipate that within the Lieb-Liniger model, β^2 is always smaller than 4π.
[39] Al. B. Zamolodchikov, Int. J. Mod. Phys. A **10**, 1125 (1995).
[40] S. N. Vergeles and V. M. Gryanik, Sov. J. Nucl. Phys. **23**, 1324 (1976).
[41] P. Weisz, Nucl. Phys. B **122**, 1 (1977); M. Karowski and P. Weisz, *ibid.* **139**, 445 (1978).
[42] F. A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory* (World Scientific, Singapore, 1992).
[43] S. Lukyanov, Mod. Phys. Lett. A **12**, 2543 (1997); arXiv:hep-th/9703190 (unpublished).
[44] S. Lukyanov, Commun. Math. Phys. **167**, 183 (1995).
[45] H. Babujian and M. Karowski, Nucl. Phys. B **620**, 407 (2002).
[46] S. Lukyanov and A. Zamolodchikov, Nucl. Phys. B **493**, 571 (1997).
[47] A. Fring, G. Mussardo, and P. Simonetti, Nucl. Phys. B **393**, 413 (1993).
[48] J. Cardy and G. Mussardo, Nucl. Phys. B **410**, 451 (1993).
[49] G. Delfino, G. Mussardo, and P. Simonetti, Nucl. Phys. B **473**, 469 (1996).
[50] We expect that these quasiclassical considerations can be qualitatively applicable in the large K limit.
[51] B. V. Chirikov, Phys. Rep. **52**, 263 (1979).
[52] V. I. Karpman, E. M. Maslov, and V. V. Solov'ev, Sov. Phys. JETP **57**, 167 (1983).
[53] C. Schori, T. Stöferle, H. Moritz, M. Köhl, and T. Esslinger, Phys. Rev. Lett. **93**, 240402 (2004).